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ROYAL AEROSPACE ESTABLISHMENT

"MINIMAL" ORBITAL DYNAMICS

by

A. J. Sarnecki

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Relations between position and velocity vectors at different points on a trajectory in a pure inverse-square field of force are derived without the use of geometrical descriptors of the orbit. An along-track 'minimal' transformation variable is found, which permits the direct integration of the equation of motion. The result is equally applicable to elliptic, hyperbolic, parabolic and rectilinear trajectories. The relationship between the transformation variable and time constitutes an archetype of Kepler's equation, conventional forms of that equation appearing as special cases. The results allow a further simplification for rectilinear motion, with the velocity used as the along-track variable. The 'minimalist' approach is also applied to the rendezvous problem: Lambert's celebrated theorem reduces to an obvious observation. Application of the theorem to the rectilinear trajectory allows the physical interpretation of parameters introduced by other authors through a purely mathematical analysis.

This is the text of a paper presented at the 38th International Astronautical Congress at Brighton in October 1987. The paper is printed here as pages 3-9, in the format required by the journal *Acta Astronautica* in which it has now been published (Vol.17, pp 881-891, 1988). The Appendix of the published paper has not been included; it amounts to a much shortened version of Ref 6 of the paper.

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Abstract

Relations between position and velocity vectors at different points on a trajectory in a pure inverse-square field of force are derived without the use of geometrical descriptors of the orbit. An along-track 'minimal' transformation variable is found, which permits the direct integration of the equation of motion. The result is equally applicable to elliptic, hyperbolic, parabolic and rectilinear trajectories. The relationship between the transformation variable and time constitutes an archetype of Kepler's equation, conventional forms of that equation appearing as special cases. The results allow a further simplification for rectilinear motion, with the velocity used as the along-track variable. The 'minimalist' approach is also applied to the rendezvous problem: Lambert's celebrated theorem reduces to an obvious observation. Application of the theorem to the rectilinear trajectory allows the physical interpretation of parameters introduced by other authors through a purely mathematical analysis.

1. Introduction

Problems involving the motion of a body, idealized as a point mass, in an inverse-square-law force field, with or without perturbations, are usually handled with the aid of a general theory of orbits, which has evolved over several centuries and includes a rich treasury of mathematical tools for analysing, describing and calculating both the instantaneous dynamical properties of the point mass and the geometrical properties of its path, the orbit.

However, the very antiquity of the dynamical problem, encountered by astronomers long before Newton's discovery of the inverse square law, has established a tradition of geometrical approaches to its solution[1-3]. Indeed, the inverse square law itself was originally deduced from the geometrical properties of observed planetary orbits. This tradition persists to this day and the illustrative nature of geometrical representation can be a help in the comprehension of many features of the dynamical behaviour. Nevertheless, the preoccupation with the geometrical properties of the orbit can sometimes obscure the physics of the motion, diverting attention from the dynamical aspects, such as energy and angular momentum, to difficulties associated with the geometric descriptors of the orbital conic, such as the parameters describing the orientation of its axes. The obscuration of the problem is particularly noticeable when the 'all-points' approach via the geometry of the orbital path is applied to problems that really involve conditions (position and velocity components) at just two points of the orbit: an 'initial' point, and the point reached at an arbitrary later (or earlier) instant.

The 'minimal' approach of this paper to the two-point problem has two main thrusts. First, the solution of the equations of motion is obtained without reference to the geometry of conic sections, though the parameters naturally entering the mathematical solution can readily be related, a posteriori, to the well-known geometric descriptors. Secondly, instead of assuming the solution of a second-order differential equation in terms of (geometrically based) trigonometric or hyperbolic functions, a universal solution is obtained by direct integration, after a suitable transformation of variable.

In section 2 a brief derivation of the second-order differential equation governing the motion is followed by a first integration, resulting in the fundamental first-order equation (the energy equation) expressed in terms of the initial position and velocity components. Section 3 presents the 'minimal' transformation needed to make that equation directly integrable. The relation between time and the along-track coordinate so introduced is derived in section 4; it appears as an archetype of Kepler's equation. The special case of rectilinear motion allows further simplification and is treated in section 5. This completes the analysis of the problem in which the conditions at some 'initial' instant are known and those at a different time are to be found.

In section 6 the results are applied to a modified two-point problem: the positions at both ends of a trajectory and the time of travel are given, and the initial (and final) velocity to achieve the rendezvous is calculated; this is usually known as the Lambert Problem and has an extensive literature. Of course, the 'minimalist' approach yields solutions which are entirely equivalent to those achieved using traditional methods: we start from the same vector differential equation and set of boundary conditions; only the methodology is different.

Finally, in section 7, the solution of the Lambert problem obtained in section 6 is compared with the formulations of Lancaster et al (Ref 4, expanded in Ref 5). The comparison allows a physical interpretation (in terms of rectilinea[illegible] parameters) of the quantities introduced in Ref 4 on a purely algebraic basis.

2. Basic dynamics

The motion of the point mass is governed by the vector differential equation of second order:

$$\frac{d^2\underline{r}}{dt^2} = -\frac{\mu\underline{r}}{r^3} , \qquad (1)$$

where $\underline{r}$ is the radius vector from the centre of attraction and μ is a constant.

If $\underline{R}$ and $\underline{Q}$ are the position and velocity vectors at an initial instant (taken as time $t = 0$), then the motion is confined to the plane through the centre of attraction (origin) and containing the vectors $\underline{R}$ and $\underline{Q}$. To establish the 'minimal' equations of motion we have to define a reference system in that plane; the obvious choice is the pair of orthogonal unit vectors $\underline{I}$ and $\underline{J}$, with $\underline{I}$ along $\underline{R}$ and the sense of $\underline{J}$ chosen to make $\underline{Q}\cdot\underline{J}$ non-negative ($\underline{J}$ is not unique when $\underline{Q}\cdot\underline{J} = 0$). Thus we can write

$$\underline{R} = R\underline{I} \quad (R > 0) \quad \text{and} \quad \underline{Q} = U\underline{I} + V\underline{J} \quad (V \geq 0). \qquad (2)$$

For the description of the position $\underline{r}$ and velocity $\underline{q}$ at an arbitrary time t, polar coordinates (r,θ) are more convenient than cartesian. Thus we have

$$\underline{r} = r(\underline{I}\cos\theta + \underline{J}\sin\theta) \qquad (3)$$

and

$$\underline{q} = \frac{dr}{dt}(\underline{I}\cos\theta + \underline{J}\sin\theta) + \left(r\frac{d\theta}{dt}\right)(\underline{J}\cos\theta - \underline{I}\sin\theta) . \qquad (4)$$

The radial and transverse components of velocity will be denoted by u and v, respectively, so that

$$\underline{q} = u(\underline{I}\cos\theta + \underline{J}\sin\theta) + v(\underline{J}\cos\theta - \underline{I}\sin\theta) , \qquad (5)$$

where $u = dr/dt$
and $v = r(d\theta/dt)$.

The differential equation governing the variation with time of the variables r, θ, u and v is obtained by differentiating (4) again and substituting in (1). This results in the equation

$$\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right](\underline{I}\cos\theta + \underline{J}\sin\theta) + \left[2\left(\frac{dr}{dt}\right)\frac{d\theta}{dt} + r\left(\frac{d^2\theta}{dt^2}\right)\right](\underline{J}\cos\theta - \underline{I}\sin\theta) = -\frac{\mu}{r^2}(\underline{I}\cos\theta + \underline{J}\sin\theta) ,$$

whence the standard results

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2} \qquad (6)$$

and

$$2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + r\left(\frac{d^2\theta}{dt^2}\right) = 0 . \qquad (7)$$

It follows from (7) that

$$r^2\left(\frac{d\theta}{dt}\right) = rv = \text{constant (angular momentum per unit mass)},$$

so that

$$v = \frac{VR}{r} \quad \text{and} \quad \frac{d\theta}{dt} = \frac{VR}{r^2} . \qquad (8)$$

Substitution in (6) yields

$$\frac{d^2r}{dt^2} - \frac{V^2R^2}{r^3} + \frac{\mu}{r^2} = 0 , \qquad (9)$$

which integrates to

$$\left(\frac{dr}{dt}\right)^2 + \frac{V^2R^2}{r^2} - \frac{2\mu}{r} = \text{constant} , = W^2 \text{, say} . \qquad (10)$$

This is the energy integral of the motion (per unit mass) multiplied by 2. The quantity W has the dimensions of velocity and is real, zero or pure imaginary, depending on the total energy of the motion (W or iW is taken as non-negative). In terms of the initial position and velocity,

$$W^2 = U^2 + V^2 - \frac{2\mu}{R} . \qquad (11)$$

The first-order differential equation relating r and t is therefore

$$\left(\frac{dr}{dt}\right)^2 = W^2 + \frac{2\mu}{r} - \frac{V^2R^2}{r^2} . \qquad (12)$$

For this differential equation to be analytically integrable, it must be possible to transform the independent variable in such a way as to make the right-hand side of (12) a perfect square. To this end we first express (12), in terms of the quadratic roots of the right-hand side in a dimensionless form, as

$$\left(\frac{dr}{dt}\right)^2 = V^2\left(P^2 - \frac{R}{r}\right)\left(\frac{R}{r} + Q^2\right) , \qquad (13)$$

$$\left.\begin{aligned} &\text{where} && V^2P^2Q^2 = W^2 , \text{ ie } VPQ = W \\ &\text{and} && V^2(P^2 - Q^2) = \frac{2\mu}{R} = U^2 + V^2 - W^2 ; \\ &\text{also} && P^2 \geq 1 \geq -Q^2 \\ &\text{and} && V^2(P^2 - 1)(1 + Q^2) = U^2 . \end{aligned}\right\} \qquad (14)$$

The quantity P is always real (and taken as positive), whereas Q has the same character as W and is related to the shape of the orbit: in hyperbolic orbits W and Q are real; in parabolic both are zero; and in elliptic orbits both are imaginary.

3. 'Minimal' transformation of variable

We attempt to turn the right-hand side of (13) into a perfect square by introducing a transformation of the form

$$\frac{R}{r} = \frac{a(\eta)}{b(\eta)} , \qquad (15)$$

where a and b are polynomials in η without a common factor.

If $Q^2 = -P^2$, the right-hand side of (13) is already a perfect square and remains such under any transformation of the form (15), including one found on the assumption that

$$P^2 + Q^2 \neq 0 . \qquad (16)$$

We therefore proceed with the assumption (16). Substitution of (15) into (13) yields

$$\left(\frac{dr}{dt}\right)^2 = V^2(P^2b - a)(a + Q^2b)/b^2 \quad . \qquad (17)$$

Now, any common factor of $(a + Q^2b)$ and $(P^2b - a)$ is also a common factor of $[P^2(a + Q^2b) - Q^2(P^2b - a)]$ and $[(a + Q^2b) + (P^2b - a)]$, ie of $(P^2 + Q^2)a$ and $(P^2 + Q^2)b$. Thus, apart from a constant, there is no common factor. It follows that $(a + Q^2b)$ and $(P^2b - a)$ must each be a perfect square. We can therefore write

$$(a + Q^2b) = [f(\eta)]^2 \quad \text{and} \quad (P^2b - a) = [g(\eta)]^2 \ , \qquad (18)$$

where f and g are polynomials in η without a common factor.

In terms of f and g ,

$$a = \frac{P^2f^2 - Q^2g^2}{P^2 + Q^2} \quad \text{and} \quad b = \frac{f^2 + g^2}{P^2 + Q^2} \quad . \qquad (19)$$

By substitution of (19) in (15) we obtain a transformation from r to the new variable η which makes the right-hand side of (13) a perfect square, whatever the form of the functions $f(\eta)$ and $g(\eta)$.

Thus

$$\frac{R}{r} = \frac{P^2f^2 - Q^2g^2}{f^2 + g^2} \qquad (20)$$

and then

$$\left(\frac{dr}{dt}\right)^2 = V^2(P^2 + Q^2)^2 \frac{f^2g^2}{(f^2 + g^2)^2} ,$$

so that

$$\frac{dr}{dt} = V(P^2 + Q^2)\frac{fg}{f^2 + g^2} \quad . \qquad (21)$$

Elimination of r between (20) and (21) yields

$$\frac{dt}{d\eta} = \frac{2R}{V}(fg' - gf')\frac{f^2 + g^2}{(P^2f^2 - Q^2g^2)^2} \ , \qquad (22)$$

where $f' = df(\eta)/d\eta \quad g' = dg(\eta)/d\eta$.

Then, without any new assumption regarding f and g , (22) may be expanded in partial fractions as

$$\frac{dt}{d\eta} = \frac{VR}{2W^2}(P^2 + Q^2)(fg' - gf')\left[\frac{1}{(Pf - Qg)^2} + \frac{1}{(Pf + Qg)^2}\right] - \frac{2\mu}{VW^2}\frac{fg' - gf'}{P^2f^2 - Q^2g^2} \quad . \qquad (23)$$

The simplest non-trivial form for $f(\eta)$, $g(\eta)$ is linear:

$$f = F + H\eta \quad \text{and} \quad g = G + K\eta \ , \qquad (24)$$

with F, G, H and K constants. Then

$$fg' - gf' = FK - GH \quad \text{(constant)}, \qquad (25)$$

and

$$\frac{fg' - gf'}{P^2f^2 - Q^2g^2} = \{(PH + QK)/(Pf + Qg) - (PH - QK)/(Pf - Qg)\}/2PQ \ , \qquad (26)$$

whilst

$$\frac{d(Pf - Qg)}{d\eta} = (PH - QK) \quad \text{and} \quad \frac{d(Pf + Qg)}{d\eta} = (PH + QK). \quad \ldots\ldots(27)$$

The constants F, G, H and K are arbitrary; the only implicit constraint on their values is that f and g should have no common factor, ie that the expression in (25) should be non-zero.

4. Kepler's equation

With the aid of (25) to (27), and using (14), the equation (23) may be rewritten as

$$dt = d\left\{\frac{\mu}{W^3}\ln\frac{Pf - Qg}{Pf + Qg} - \frac{VR}{2W^2}(P^2 + Q^2)(FK - GH) \times \left[\frac{1/(PH - QK)}{Pf - Qg} + \frac{1/(PH + QK)}{Pf + Qg}\right]\right\} \ . \quad \ldots\ldots(28)$$

This may be considered the archetype of Kepler's equation. When W (and hence Q) is pure imaginary, the logarithm in (28) is multi-valued, consecutive values differing by $2\pi i$; thus identical values of the motion parameters occur at time intervals of $2\pi\mu i/W^3$. This implies that the orbit is periodic, with period T given by

$$T = \frac{2\pi\mu i}{W^3} = \frac{2\pi\mu}{(iW)^3} \quad . \qquad (29)$$

Identifying the multivalued logarithm as i times an angle E , we can eliminate η in terms of E . Then we get

$$dt = d\left\{\frac{\mu}{(iW)^3}E - \frac{VR}{2(iW)^2}\frac{P^2 - (iQ)^2}{P(iQ)}\sin E\right\} \ , \qquad (30a)$$

which is a more familiar form of Kepler's equation, with E the eccentric anomaly, and with the mean motion and eccentricity expressed in terms of V, R, W, P and Q . When W (and hence Q) is real, on the other hand, the logarithm in (28) is single-valued and can be identified as the hyperbolic equivalent of eccentric anomaly (usually denoted by B); elimination of η leads to the hyperbolic analogue of Kepler's equation

$$dt = d\left\{\frac{VR}{2W^2}\frac{P^2 + Q^2}{PQ}\sinh B - \frac{\mu}{W^3}B\right\} \quad . \qquad (30b)$$

The elliptic and hyperbolic forms of equation (30) both require the explicit determination of the parameters P and Q , as defined in (13). This is equivalent to finding the pericentre of the orbit, which is a geometrical concept extraneous to the 'minimal' problem of relating the position and velocity at a 'final' instant to those at an 'initial' instant. To tackle the problem without calculating P and Q , we can introduce a constraint on F, G, H and K , so as to set $\eta = 0$ at $t = 0$, when $r = R$ and $u = U$. Then, according to (20) and (21), we must have

$$P^2F^2 - Q^2G^2 = F^2 + G^2 \quad \text{and} \quad V(P^2 + Q^2)FG = U(F^2 + G^2) \ . \quad \ldots\ldots(31)$$

It follows that

$$\frac{F^2}{1 + Q^2} = \frac{G^2}{P^2 - 1} = \frac{F^2 + G^2}{P^2 + Q^2} = \frac{VFG}{U} \ , \qquad (32)$$

so that, in terms of F and G ,

$$P^2 = (UG + VF)/VF \quad \text{and} \quad Q^2 = (UF - VG)/VG \quad . \qquad (33)$$

In view of (14) and (33), F and G are constrained by

$$UV(F^2 - G^2) = (V^2 + W^2 - U^2)FG \quad . \tag{34}$$

The denominator in the right-hand sides of (20) and (21) is $f^2 + g^2$; expressed in terms of η ,

$$f^2 + g^2 = (F^2 + G^2) + 2(FH + GK)\eta + (H^2 + K^2)\eta^2 \quad . \tag{35}$$

So far H and K are unconstrained; it is convenient to choose them so that the linear term in (35) vanishes. Then we put

$$H = -\frac{GK}{F} \ , \tag{36}$$

and get, with the aid of (33),

$$\begin{aligned} f^2 + g^2 &= (F^2 + G^2)\,[1 + K^2\eta^2/F^2] \ , \\ P^2f^2 - Q^2g^2 &= (F^2 + G^2)\,[(1 - KU\eta/VF)^2 - (KW\eta/VF)^2] \ , \\ V(P^2 + Q^2)fg &= (F^2 + G^2)\,[U + (V^2 + W^2 - U^2)K\eta/VF \\ &\qquad - UK^2\eta^2/F^2] \ . \end{aligned} \tag{37}$$

On substitution in (20) and (21) we get

$$R/r = [(1 - KU\eta/VF)^2 - (KW\eta/VF)^2] \,/\, [1 + K^2\eta^2/F^2]$$

and

$$\frac{dr}{dt} = [U + (V^2 + W^2 - U^2)K\eta/VF - UK^2\eta^2/F^2] \,/\, [1 + K^2\eta^2/F^2] \ ,$$

and then, on denoting $K\eta/VF$ by χ ,

$$r = \frac{R(1 + V^2\chi^2)}{[(1 - U\chi)^2 - (W\chi)^2]} \tag{38}$$

and

$$u = \frac{dr}{dt} = \frac{U + (V^2 + W^2 - U^2)\chi - UV^2\chi^2}{1 + V^2\chi^2} \quad . \tag{39}$$

Then, on eliminating r between (38) and (39), we obtain

$$\begin{aligned} \frac{dt}{d\chi} &= 2R\,\frac{1 + V^2\chi^2}{[1 - (U + W)\chi]^2\,[1 - (U - W)\chi]^2} \\ &= \frac{\mu}{W^3}\left[\frac{U - W}{1 - U\chi + W\chi} - \frac{U + W}{1 - U\chi - W\chi}\right] + \frac{R}{2W^2} \,\times \\ &\quad \times\left[\frac{V^2 + (U + W)^2}{(1 - U\chi - W\chi)^2} + \frac{V^2 + (U - W)^2}{(1 - U\chi + W\chi)^2}\right] \ , \end{aligned} \tag{40}$$

so that

$$\begin{aligned} dt = d\Bigg\{&\frac{\mu}{W^3}\ln\frac{1 - U\chi - W\chi}{1 - U\chi + W\chi} + \frac{R}{2W^2} \,\times \\ &\times\left[\frac{V^2 + (U + W)^2}{(U + W)(1 - U\chi - W\chi)} + \frac{V^2 + (U - W)^2}{(U - W)(1 - U\chi + W\chi)}\right]\Bigg\} \end{aligned} \tag{41}$$

and therefore, since $t = 0$ when $\chi = 0$, the 'minimal' version of Kepler's equation is

$$\frac{W^3 t}{\mu} = \ln\frac{1 - U\chi - W\chi}{1 - U\chi + W\chi} + \frac{RW\chi}{2\mu}\left[\frac{V^2 + (U + W)^2}{1 - U\chi - W\chi} + \frac{V^2 + (U - W)^2}{1 - U\chi + W\chi}\right] \ . \tag{42}$$

Just as for other forms of Kepler's equation, when t is given, the along-track parameter χ has to be found from (42); then the quantities r, u, v and θ can be determined as follows:

r is given by (38) ;

u is given by (39) ;

substitution of (38) in (8) yields

$$v = V[(1 - U\chi)^2 - (W\chi)^2] \,/\, (1 + V^2\chi^2) \tag{43}$$

and

$$\begin{aligned} \theta &= \int (VR/r^2)\,dt = \int [2(1 + V^2\chi^2)]\,V d\chi \\ &= 2 \arctan V\chi \ . \end{aligned} \tag{44}$$

Thus $V\chi$ is identified as the tangent of half the angle turned by the radius vector from the initial position.

In vectorial form the results can be written as

$$\begin{aligned} \underline{r} &= r[(1 - V^2\chi^2)\underline{I} + 2V\chi\underline{J}] \,/\, (1 + V^2\chi^2) \\ &= R[(1 - V^2\chi^2)\underline{I} + 2V\chi\underline{J}] \,/\, [(1 - U\chi)^2 - (W\chi)^2] \end{aligned} \tag{45}$$

and

$$\begin{aligned} \underline{q} &= \frac{\{u[(1 - V^2\chi^2)\underline{I} + 2V\chi\underline{J}] + v[(1 - V^2\chi^2)\underline{J} - 2V\chi\underline{I}]\}}{1 + V^2\chi^2} \\ &= \{U + (W^2 - U^2 - V^2)\chi \,/\, (1 + V^2\chi^2)\}\underline{I} \\ &\quad + \{V[1 + (W^2 - U^2)\chi^2] / (1 + V^2\chi^2)\}\underline{J} \quad . \end{aligned} \tag{46}$$

These formulae apply for all values of U, V and (non-zero) R . However, when $W = 0$ (zero total energy, orbit parabolic), equation (42) takes the form $0 = 0$ and has to be expressed differently, eg by expanding both sides in powers of W and comparing the leading terms, allowing for the constraint between U, V and R expressed by (11). The result is

$$t = \frac{2R\chi}{(1 - U\chi)^2} + \frac{4\mu\chi^3}{3(1 - U\chi)^3} \quad . \tag{47}$$

5. Rectilinear motion

Equations (38) to (47) require no modification when $V = 0$, though the right-hand sides of (43) and (44) become identically zero, expressing the physical fact that the motion is confined to a straight line. Furthermore, with the vanishing of the terms containing $V^2\chi^2$ in (39), that equation becomes linear and allows the elimination of χ in terms of the (radial) velocity u :

$$\chi = \frac{u - U}{W^2 - U^2} \quad . \tag{48}$$

Substitution in (42) and (38) then yields (with $V = 0$)

$$\begin{aligned} t = \frac{\mu}{W^3}\Bigg\{&\ln\frac{(U + W)(u - W)}{(U - W)(u + W)} - \frac{RW}{2\mu}(u - U) \,\times \\ &\times\left[\frac{U + W}{u - W} + \frac{U - W}{u + W}\right]\Bigg\} \end{aligned} \tag{49}$$

and

$$r = \frac{R(U^2 - W^2)}{u^2 - W^2} \quad . \tag{50}$$

In rectilinear motion, equation (11), defining W, reduces to

$$W^2 = U^2 - \frac{2\mu}{R} \; ; \qquad (51)$$

and the equations (50) and (51) may be combined as

$$r(u^2 - W^2) = R(U^2 - W^2) = 2\mu \; . \qquad (52)$$

Although this cannot occur in the real world, *if the attracting central body is reduced to a* point, then the pericentre of the rectilinear motion occurs at $r = 0$. The velocity u is infinite there and changes sign from negative to positive as the point mass 'bounces off' the centre of attraction. The time from the initial point to the centre is found from (49) by taking limits as u tends to infinity:

$$t_p = \frac{\mu}{W^3}\left(\ln\frac{U + W}{U - W} - \frac{RWU}{\mu}\right) \; . \qquad (53)$$

In the further limit as $W \to 0$ (rectilinear parabola), we have

$$\left.\begin{aligned} \chi &= \frac{U - u}{U^2} \; , \\ ru^2 &= RU^2 = 2\mu \; , \\ t &= \frac{4}{3}\mu\left(\frac{1}{u^3} - \frac{1}{U^3}\right) \end{aligned}\right\} \qquad (54)$$

and

$$t_p = -\frac{4\mu}{3U^3} = -\frac{2R}{3U} \; . \qquad (55)$$

6. The Lambert Problem and Lambert's Theorem

So far we have considered the 'two-point' problem with the initial position and velocity both given. The results can be applied also to the problem in which it is the initial and final positions that are given, together with the time interval, and the velocities at the end points are to be found; this is known as the Lambert Problem (or the rendezvous problem).

The equations (38) to (47) continue to apply, but now r and θ (as well as R) are given, whilst U and V (and hence W) are unknown, as also is the transformed variable χ. With t given, equation (42) contains four unknowns (χ, U, V *and* W) *constrained by (11), (38) and (44).*

The complexity of the problem (four equations in four unknowns) is reduced if the geometry of the end-point triangle (formed by the two end points and the centre of attraction) is specified not by the two sides and included angle (R, r and θ) but by the three sides, viz R, r and c, where c is the distance between the end points, so that

$$c^2 = R^2 + r^2 - 2Rr\cos\theta$$

and

$$V^2\chi^2 = \tan^2\tfrac{1}{2}\theta = \frac{1 - \cos\theta}{1 + \cos\theta} = \frac{c^2 - (R - r)^2}{(R + r)^2 - c^2} \; . \qquad (56)$$

In terms of R, r and c, the four equations in χ, U, V and W, viz (42), (38), (56) and (11), can be written as

$$t = \frac{\mu}{W^3}\left\{\ln\frac{1 - U\chi - W\chi}{1 - U\chi + W\chi} + \frac{2W\chi}{(1 - U\chi)^2 - (W\chi)^2} \times \left[\frac{R}{\mu}W^2 + 1 - U\chi\right]\right\} \; , \qquad (57)$$

$$r[(1 - U\chi)^2 - (W\chi)^2] = R(1 + V^2\chi^2) \; , \qquad (58)$$

$$[(R + r)^2 - c^2]V^2\chi^2 = c^2 - (R - r)^2 \qquad (59)$$

and

$$R(U^2 + V^2 - W^2) = 2\mu \; . \qquad (60)$$

With the aid of (58) we can eliminate V^2 from (59) and (60), getting

$$[(R + r)^2 - c^2][(1 - U\chi)^2 - (W\chi)^2] = 4R^2 \qquad (61)$$

and

$$r[(1 - U\chi)^2 - (W\chi)^2] + [R(U^2 - W^2) - 2\mu]\chi^2 = R \; . \qquad (62)$$

This leaves (57), (61) and (62) as three equations in the three unknowns χ, U and W. However, the equations can be rewritten so that the unknowns appear only in the combinations

$$\left.\begin{aligned} & W \; , \\ [(1 - U\chi)/R] \; , &= X \; , \text{ say} \\ \text{and} \quad [W\chi/R] \; , &= Y \; , \text{ say.} \end{aligned}\right\} \qquad (63)$$

The coefficients in the equations depend on the known values of μ, t, c and $(R + r)$, but are independent of $(R - r)$; thus

$$\frac{\mu}{W^3}\left\{\ln\frac{X - Y}{X + Y} + \frac{2Y}{X^2 - Y^2}\left[\frac{W^2}{\mu} + X\right]\right\} = t \; , \qquad (64)$$

$$[(R + r)^2 - c^2][X^2 - Y^2] = 4 \qquad (65)$$

and

$$(R + r)W^2[X^2 - Y^2] - 2\mu Y^2 = 2W^2X \; . \qquad (66)$$

It follows that the quantities W, X and Y in (63) depend only on μ, t, c and $R + r$, and not on $R - r$. This proves Lambert's Theorem, which states that, with μ and energy (ie W) given, the time of transit t depends only on $R + r$ and c.

Lambert's Theorem permits the definition of equivalence classes of triangles (formed by the *centre of attraction and the two end points of the* trajectory) for which the relation connecting t, W, X and Y is identical for the members of a class; all such members have the same $R + r$ and c but are distinguished by the differing values of $R - r$. This is considered further by Gooding[6].

In particular, each equivalence class includes a pair of degenerate triangles for which

$$|R - r| = c \; ,$$

ie such that the two end points and the centre of attraction are collinear. The variation of W, X and Y with time can easily be found for the degenerate triangle (in which the motion is rectilinear), but the results so obtained apply to all members of the particular equivalence class. We distinguish the outer and inner end points of the equivalent rectilinear motion by the suffixes 1 and

2, retaining the unsuffixed notation for the two-dimensional original problem. Thus the two sets of radii and velocities are related by

$$2R_1 = R + r + c \;, \qquad 2R_2 = R + r - c \qquad (67)$$

and, in view of (60),

$$R_1\left(U_1^2 - W^2\right) = R_2\left(U_2^2 - W^2\right) = 2\mu = R(U^2 + V^2 - W^2) = r(u^2 + v^2 - W^2) \;. \qquad ......(68)$$

From (49), the equation relating transit time to the other quantities is

$$t = \frac{\mu}{W^3}\left\{\ln\frac{(U_1 + W)(U_2 - W)}{(U_1 - W)(U_2 + W)} + \frac{W}{\mu}\,[R_2U_2 - R_1U_1]\right\}, \qquad (69)$$

where the skew-symmetric form of the second right-hand term has been obtained with the aid of (68).

A numerical solution for U_1, U_2 and W can now be obtained using one of the unknown velocities as the iteration variable and calculating the others with the aid of (68). An accurate computing procedure for the solution, using a dimensionless form of U_1 as the iteration variable, is described by Gooding[6].

Having found the values of U_1, U_2 and W for the given transit time t, we can determine the other parameters in (63), since, from (48)

$$X = [1 - U_1(U_1 - U_2)/(U_1^2 - W^2)]/R_1 = [U_1U_2 - W^2]/2\mu \qquad ......(70)$$

and

$$Y = W[(U_1 - U_2)/(U_1^2 - W^2)]/R_1 = W(U_1 - U_2)/2\mu \;, \qquad ......(71)$$

and therefore, *for the individual members of the equivalence class,*

$$\chi = \frac{RY}{W} \qquad (72)$$

and

$$U = W[1 - RX]/RY \;, \qquad (73)$$

whilst from (36) and (72) it follows that

$$V = \left|\frac{W}{RY}\right|\left(\frac{c^2 - (R - r)^2}{(R + r)^2 - c^2}\right)^{\frac{1}{2}} . \qquad (74)$$

Also, from the symmetry of the geometry (or from a manipulation of previously derived formulae) it *follows that*

$$u = - W[1 - rX]/rY \qquad (75)$$

and

$$v = \left|\frac{W}{rY}\right|\left(\frac{c^2 - (R - r)^2}{(R + r)^2 - c^2}\right)^{\frac{1}{2}} . \qquad (76)$$

This completes the solution of the Lambert Problem.

7. The Lancaster-Blanchard-Devaney method

Lancaster, Blanchard and Devaney[4] ("L-B-D") developed a method for solving the Lambert problem in a coherent fashion, by starting from the classical formulae which are different for elliptic and hyperbolic trajectories (the parabolic trajectory being *a transitional case*) and applying suitable (and different) transformations of variable to arrive at a single algorithm for both types of orbit. The transformations are introduced as mathematical tools of convenience and no physical interpretation is offered by the authors for the end results. In fact, the L-B-D parameters can be interpreted in terms of the Lambert-equivalent rectilinear motion, as discussed in the present paper. As the L-B-D parameters are all dimensionless, we first have to express our formulae in a dimensionless form. To this end we must define a scale length and scale velocity related to the geometry of the end-point triangle (but common to a whole equivalence class), independent of the particular time of transit (which is the input variable to the problem). These scales may be defined as the radius and the escape velocity at the outer point of the Lambert-equivalent rectilinear triangle, ie

$$\left.\begin{aligned} &\text{length scale} = R_1 \\ \text{and}\quad &\text{velocity scale} = \sqrt{(2\mu/R_1)} \;, \\ \text{whence}\quad &\text{time scale} = \sqrt{(R_1^3/2\mu)} \;. \end{aligned}\right\} \qquad (77)$$

The dimensionless versions of our formulae (68), (69) and (72) to (76) are

$$U_1^2 - W^2 = R_2\left[U_2^2 - W^2\right] = 1 \;, \qquad (78)$$

$$t = \frac{1}{W^3}\left(\tfrac{1}{2}\ln\frac{[U_1 + W][U_2 - W]}{[U_1 - W][U_2 + W]} + W[R_2U_2 - U_1]\right) \;, \qquad (79)$$

$$\chi = R[U_1 - U_2] \qquad (80)$$

$$U = \left[\frac{1}{R} + W^2 - U_1U_2\right]/[U_1 - U_2] \;, \qquad (81)$$

$$V = \frac{1}{R|U_1 - U_2|}\left\{\frac{c^2 - (R - r)^2}{(R + r)^2 - c^2}\right\}^{\frac{1}{2}} , \qquad (82)$$

$$u = \left[\frac{1}{r} + W^2 - U_1U_2\right]\Big/[U_2 - U_1] \qquad (83)$$

and

$$v = \frac{1}{r|U_1 - U_2|}\left\{\frac{c^2 - (R - r)^2}{(R + r)^2 - c^2}\right\}^{\frac{1}{2}} . \qquad (84)$$

The L-B-D parameters are related to those of the present paper according to the following table:

L-B-D notation		Present notation
	Lengths	
s		R_1
c		c
	Dimensionless quantities	
K $(= q^2)$		R_2
T		$2t$
E		W^2
x		$-U_1$
y (elliptic)		iW
(hyperbolic)		W
z/q		$-U_2$
x-qz		$R_2[U_2 - U_1]$

d	(elliptic)	$-\frac{1}{2}i \ln \frac{[U_1 + W][U_2 - W]}{[U_1 - W][U_2 + W]}$
	(hyperbolic)	$-\frac{1}{2} \ln \frac{[U_1 + W][U_2 - W]}{[U_1 - W][U_2 + W]}$
f/q	(elliptic)	$iW[U_1 - U_2]$
	(hyperbolic)	$W[U_1 - U_2]$
g/q		$U_1U_2 - W^2$

The multivalued character of the logarithmic term, when W is imaginary, is reflected in the indeterminate parameter m of the L-B-D analysis. No physical interpretation is offered for the angular parameters α, β, γ and δ, which are related[2,3] to the geometry of the elliptic or hyperbolic trajectory. A final comment concerns the separation of the expression for the transit time into two terms described by the L-B-D function ϕ (and used only for near-parabolic trajectories). The two terms correspond, in the rectilinear equivalent motion, to the times taken from either end of the trajectory to the instant of 'bouncing' off the centre of attraction, the pericentre of the rectilinear motion.

8. Conclusions

The 'minimalist' approach to orbital dynamics leads naturally to various basic results, in particular as regards the prediction of position and velocity at a later time when initial conditions are given, and in the solution of Lambert's rendezvous problem. The proof of Lambert's Theorem becomes almost trivial. Furthermore, the elegant formulation of Lancaster, Blanchard and Devaney is given a physical interpretation.

This approach, applied to the unperturbed inverse-square-law field of force, bypasses the classical analyses anchored to the geometrical properties of the orbit, and avoids the complications associated with the orientation of the orbital conic.

Orbital motion of a point mass is described by a three-dimensional vector differential equation of the second order, and therefore requires six parameters (orbital elements) to describe it fully. Five of these may be used to describe the path in space, and the sixth to relate it to absolute time (as by stating the time at pericentre or the crossing of a reference plane). The classical set of spatial elements[1-3] consists of two describing the shape and size of the orbit (a,e) and three angles giving the orientation of the axes of the orbit (i,Ω,ω). An alternative spatial set is provided by the five independent components of the (mutually orthogonal) angular momentum and eccentricity vectors, which can be defined in terms of the position and velocity vectors. In principle it does not matter which six elements are used to describe the orbit, provided they are not subject to an internal constraint. In practice some choices make the problem ill-conditioned (eg pericentre-related parameters in near-circular orbits, or node-related parameters in near-equatorial orbits). Additional difficulties may arise in numerical work, with loss of accuracy when small differences between large quantities have to be calculated. There can therefore be no hard-and-fast rule which would state that a particular set of orbital elements is better than all others in all circumstances. The success achieved here with the direct relationships between the position and velocity vectors at two particular points, supplemented only by the velocity W, suggests that perturbation effects might be expressible in terms of the (slow) variation of the position and velocity at a selected epoch.

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17. Abstract

Relations between position and velocity vectors at different points on a trajectory in a pure inverse-square field of force are derived without the use of geometrical descriptors of the orbit. An along-track 'minimal' transformation variable is found, which permits the direct integration of the equation of motion. The result is equally applicable to elliptic, hyperbolic, parabolic and rectilinear trajectories. The relationship between the transformation variable and time constitutes an archetype of Kepler's equation, conventional forms of that equation appearing as special cases. The results allow a further simplification for rectilinear motion, with the velocity used as the along-track variable. The 'minimalist' approach is also applied to the rendezvous problem: Lambert's celebrated theorem reduces to an obvious observation. Application of the theorem to the rectilinear trajectory allows the physical interpretation of parameters introduced by other authors through a purely mathematical analysis. Great Britain (JJ) JE

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